

A survey on the generalized connectivity of graphs*

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Abstract

The generalized k -connectivity $\kappa_k(G)$ of a graph G was introduced by Chartrand et al. in 1984. As a natural counterpart of it, we introduced the concept of generalized edge-connectivity $\lambda_k(G)$, recently. In this paper we summarize the known results on generalized connectivity and generalized edge-connectivity. The paper is divided into the following five categories: the generalized (edge-) connectivity of some graph classes, sharp bounds of $\kappa_k(G)$ and $\lambda_k(G)$, graphs with large generalized (edge-) connectivity, graph operations, extremal problems, and algorithms and computational complexity. It also contains some conjectures and open problems for further studies.

Keywords: connectivity, internally disjoint trees, edge-connectivity, edge-disjoint trees, generalized connectivity, generalized edge-connectivity, algorithm and complexity.

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1 Introduction

In this introductory section, we will give both theoretical and practical motivation for introducing the concept of generalized (edge-) connectivity of graphs. Some definitions on graph theory are also given.

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1.1 Motivation and definition

Connectivity is one of the most basic concepts of graph-theoretic subjects, both in a combinatorial sense and an algorithmic sense. There are many elegant and powerful results on connectivity in graph theory. The *connectivity* $\kappa(G)$ of a graph G is defined as the minimum cardinality of a set Q of vertices of G such that $G - Q$ is disconnected or trivial. A well-known theorem of Whitney provides an equivalent definition of connectivity. For each 2-subset $S = \{u, v\}$ of $V(G)$, let $\kappa(S)$ denote the maximum number of internally disjoint uv paths in G . Then $\kappa(G) = \min\{\kappa(S)\}$, where the minimum is taken over all 2-subsets S of $V(G)$.

The generalized connectivity of a graph G , introduced by Chartrand et al. in [3], is a natural and nice generalization of the concept of (vertex-) connectivity. Let G be a nontrivial connected graph of order n and let k be an integer with $2 \leq k \leq n$. For a set S of k vertices of G , let $\kappa(S)$ denote the maximum number ℓ of edge-disjoint trees T_1, T_2, \dots, T_ℓ in G such that $V(T_i) \cap V(T_j) = S$ for every pair i, j of distinct integers with $1 \leq i, j \leq \ell$ (Note that these trees are vertex-disjoint in $G \setminus S$). A collection $\{T_1, T_2, \dots, T_\ell\}$ of trees in G with this property is called a *set of internally disjoint trees connecting S* . The *generalized k -connectivity*, denoted by $\kappa_k(G)$, of G is then defined as $\kappa_k(G) = \min\{\kappa(S) | S \subseteq V(G) \text{ and } |S| = k\}$. Thus, $\kappa_2(G) = \kappa(G)$. Set $\kappa_k(G) = 0$ when G is disconnected.

As a natural counterpart of the generalized connectivity, we introduced the concept of generalized edge-connectivity recently in [20]. Let $\lambda(S)$ denote the maximum number ℓ of pairwise edge-disjoint trees T_1, T_2, \dots, T_ℓ in G such that $V(T_i) \supseteq S$ for every $1 \leq i \leq \ell$ (Note that these (edge-disjoint) trees are *Steiner trees*). Then the *generalized k -edge-connectivity* $\lambda_k(G)$ of G is defined as $\lambda_k(G) = \min\{\lambda(S) | S \subseteq V(G) \text{ and } |S| = k\}$. Thus $\lambda_2(G) = \lambda(G)$. Set $\lambda_k(G) = 0$ when G is disconnected.

We must point out that the generalized connectivity and the generalized edge-connectivity of a graph are different in general. The following graph H in Figure 1.1 (a) is an example for which $\kappa_3(H) = 1$ but $\lambda_3(H) = 2$.

The generalized edge-connectivity is related to an important problem, which is called the *Steiner Tree Packing Problem*. For a given graph G and $S \subseteq V(G)$, this problem asks to find set of maximum size of edge-disjoint Steiner trees connecting S of G . There are two differences between the Steiner Tree Packing Problem and the generalized edge-connectivity: One is that the former considers graphs containing multiple edges but the latter studies simple graphs; the other is that the Steiner Tree Packing Problem studies local properties of graphs since S is given beforehand, but

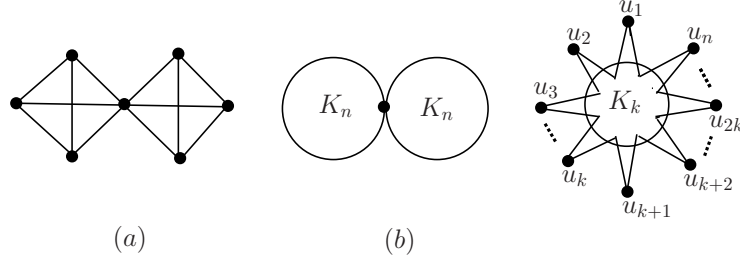


Figure 1.1 (a) $\kappa_3(H) = 1, \lambda_3(H) = 2$; (b) $\kappa_k(G) = 1, \lambda_k(G) = n - \lceil \frac{k}{2} \rceil$; (c) Graph G with $\kappa_k(G) = \lambda_k(G) = \kappa(G) = \lambda(G) = \delta(G) = k$.

the generalized edge-connectivity focuses on global properties of graphs since it first needs to compute the maximum number $\lambda(S)$ of edge-disjoint trees connecting S and then S runs over all k -subsets of $V(G)$ to get the minimum value of $\lambda(S)$.

The problem for $S = V(G)$ is called the *Spanning Tree Packing Problem* (Note that the Steiner Tree Packing Problem is a generalization of the Spanning Tree Packing Problem). For any graph G of order n , the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in G . For the spanning tree packing number, Palmer gave a good survey, see [5]. For simple graphs, the *STP* number of a graph G is just $\kappa_n(G) = \lambda_n(G)$ (For $k = n$, these internally disjoint trees and edge-disjoint trees are edge-disjoint spanning trees).

In addition to being natural combinatorial measures, the generalized connectivity and generalized edge-connectivity can be motivated by their interesting interpretation in practice as well as theoretical consideration.

From a theoretical perspective, both extremes of this problem are fundamental theorems in combinatorics. One extreme of the problem is when we have two terminals. In this case internally (edge-) disjoint trees are just internally (edge-) disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case internally disjoint trees and edge-disjoint trees are just spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem.

Theorem 1.1 (*Nash-Williams [6], Tutte [7]*) *A multigraph G contains a system of k edge-disjoint spanning trees if and only if*

$$\|G/\mathcal{P}\| \geq k(|\mathcal{P}| - 1)$$

holds for every partition \mathcal{P} of $V(G)$, where $\|G/\mathcal{P}\|$ denotes the number of edges in G between distinct blocks of \mathcal{P} .

It is an easy exercise to show, by Theorem 1.1, that if a graph G is $2k$ -edge-connected then it has k edge-disjoint spanning trees. Kriesell [8] conjectured that this corollary can be generalized for Steiner trees.

Conjecture 1.2 (Kriesell [8]) *If a set S of vertices of G is $2k$ -edge-connected (See Section 1.2 for the definition), then there is a set of k edge-disjoint Steiner trees in G .*

Motivated by this conjecture, the Steiner Tree Packing Problem has obtained wide attention and many results have been worked out, see [8, 9, 10, 11, 12]. In [20], we built the relation between the Steiner Tree Packing Problem and the generalized edge-connectivity.

The generalized edge-connectivity and the Steiner Tree Packing Problem have applications in *VLSI* circuit design, see [13, 14, 15]. In this application, a Steiner tree is needed to share an electronic signal by a set of terminal nodes. Another application, which is our primary focus, arises in the Internet Domain. Imagine that a given graph G represents a network. We choose arbitrary k vertices as nodes. Suppose one of the nodes in G is a *broadcaster*. All other nodes are either *users* or *routers* (also called *switches*). The broadcaster wants to broadcast as many streams of movies as possible, so that the users have the maximum number of choices. Each stream of movie is broadcasted via an edge-disjoint tree connecting all the users and the broadcaster. So, in essence we need to find the maximum number Steiner trees connecting all the users and the broadcaster, namely, we want to get $\lambda(S)$, where S is the set of the k nodes. Clearly, it is a Steiner tree packing problem. Furthermore, if we want to know whether for any k nodes the network G has above properties, then we need to compute $\lambda_k(G) = \min\{\lambda(S)\}$ in order to prescribe the reliability and the security of the network.

1.2 Notation and terminology

All graphs considered in this paper are undirected, finite and simple. We refer to book [1] for graph theoretical notation and terminology not described here. For a graph G , let $V(G)$, $E(G)$, $|G|$, $e(G)$, $L(G)$ and \overline{G} denote the set of vertices, the set of edges, the order, the size, the line graph and the complement of G , respectively. As usual, the *union*, denoted by $G \cup H$, of two graphs G and H is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$. The disjoint union of k copies of the same

graph G is denoted by kG . The *join* $G_1 \vee G_2$ of G_1 and G_2 is obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to every vertex of G_2 . For $S \subseteq V(G)$, we denote $G \setminus S$ the subgraph by deleting the vertices of S together with the edges incident with them from G . If $S = \{v\}$, we simply write $G \setminus v$ for $G \setminus \{v\}$. If S is a subset of vertices of a graph G , the subgraph of G induced by S is denoted by $G[S]$. If M is the edge subset of G , then $G \setminus M$ denote the subgraph by deleting the edges of M . If M is a subset of edges of a graph G , the subgraph of G induced by M is denoted by $G[M]$. If $M = \{e\}$, we simply write $G \setminus e$ for $G \setminus \{e\}$. We denote by $E_G[X, Y]$ the set of edges of G with one end in X and the other end in Y . If $X = \{x\}$, we simply write $E_G[x, Y]$ for $E_G[\{x\}, Y]$. Let $K_{a,b}$ denote a complete bipartite graph. We denote by K_b^a a complete a -partite graph which contains b vertices in each part. A complete equipartition 3-partite graph is a complete 3-partite graph in which every part contains exactly b vertices, denoted by K_b^3 . The *Cartesian product* (also called square product) of two graphs G and H , written as $G \square H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices (u, v) and (u', v') are adjacent if and only if $u = u'$ and $(v, v') \in E(H)$, or $v = v'$ and $(u, u') \in E(H)$. A graph G is k -regular if $d(v) = k$ for all $v \in V(G)$. A 3-regular graph is called *cubic*.

For distinct vertices x, y in G , let $\lambda(x, y)$ denote the local edge-connectivity of x and y . A subset $S \subseteq V(G)$ is called t -edge-connected, if $\lambda(x, y) \geq t$ for all $x \neq y$ in S . A k -connected graph G is *minimally k -connected* if the graph $G - e$ is not k -connected for any edge of G . A graph is minimal for $\kappa_k = \ell$ if the generalized k -connectivity of G is ℓ but the generalized k -connectivity of $G - e$ is less than ℓ for any edge e of G . For $X = \{x_1, x_2, \dots, x_k\}$ and $Y = \{y_1, y_2, \dots, y_k\}$, an XY -linkage is defined as a set of k vertex-disjoint paths $x_i P_i y_i$, $1 \leq i \leq k$. The *Linkage Problem* is the problem of deciding whether there exists an XY -linkage for given sets X and Y .

A *decision problem* is a question whose answer is either yes or no. Such a problem belongs to the class \mathcal{P} if there is a polynomial-time algorithm that solves any instance of the problem in polynomial time. It belongs to the class \mathcal{NP} if, given any instance of the problem whose answer is yes, there is a certificate validating this fact, which can be checked in polynomial time; such a certificate is said to be *succinct*. It is immediate from these definitions that $\mathcal{P} \subseteq \mathcal{NP}$, inasmuch as a polynomial-time algorithm constitutes, in itself, a succinct certificate. A *polynomial reduction* of a problem P to a problem Q is a pair of polynomial-time algorithms, one of which transforms each instance I of P to an instance J of Q , and the other of which transforms a solution for the instance J to a solution for the instance I . If such a reduction exists, we say that P is *polynomially reducible* to Q , and write $P \preceq Q$. A problem P in \mathcal{NP} is \mathcal{NP} -complete

if $P' \preceq P$ for every problem P' in \mathcal{NP} . The class of \mathcal{NP} -complete problems is denoted by \mathcal{NPC} . Even if every problem in \mathcal{NP} is polynomially reducible to some problem Q , we are unable to argue that $Q \in \mathcal{NP}$.

3-DIMENSIONAL MATCHING (3-DM) Given three sets U , V , and W of equal cardinality, and a subset T of $U \times V \times W$, decide whether there is a subset M of T with $|M| = |U|$ such that whenever (u, v, w) and (u', v', w') are distinct triples in M , then $u \neq u'$, $v \neq v'$, and $w \neq w'$?

BOOLEAN 3-SATISFIABILITY (3-SAT) Given a boolean formula ϕ in conjunctive normal form with three literals per clause, decide whether ϕ is satisfiable?

2 Results for some graph classes

The following two observations are easily seen.

Observation 2.1 *If G is a connected graph, then $\kappa_k(G) \leq \lambda_k(G) \leq \delta(G)$.*

Observation 2.2 *If H is a spanning subgraph of G , then $\kappa_k(H) \leq \kappa_k(G)$ and $\lambda_k(H) \leq \lambda_k(G)$.*

Chartrand, Kapoor and Lesniak in [4] proved that if G is a complete 3-partite graph $K_{3,4,5}$, then $\kappa_3(G) = 6$. They also got the exact value of generalized k -connectivity for complete graph K_n .

Theorem 2.3 [4] *For every two integers n and k with $2 \leq k \leq n$,*

$$\kappa_k(K_n) = n - \lceil k/2 \rceil.$$

In [21], Li, Mao and Sun obtained the explicit value for $\lambda_k(K_n)$. One may not expect that it is the same as $\kappa_k(K_n)$.

Theorem 2.4 [21] *For every two integers n and k with $2 \leq k \leq n$,*

$$\lambda_k(K_n) = n - \lceil k/2 \rceil.$$

From Theorems 2.3 and 2.4, we get that $\lambda_k(G) = \kappa_k(G)$ for a complete graph G . However, this is a very special case. Actually, $\lambda_k(G) - \kappa_k(G)$ could be very large. For example, let G be a graph obtained from two copies of the complete graph K_n by identifying one vertex in each of them, see Figure 1.1 (b). Then $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$, but $\kappa_k(G) = 1$.

Okamoto and Zhang [27] investigated the generalized k -connectivity of a regular complete bipartite graph $K_{a,a}$. Naturally, one may ask whether we can compute the value of generalized k -connectivity of a complete bipartite graph $K_{a,b}$, or even complete multipartite graphs. S. Li, W. Li and X. Li [17] devoted to solving such problems.

Actually, Palmer obtained the number of edge-disjoint spanning trees of $K_{a,b}$ in 2001. He also gave the STP numbers of some graph classes. For more details we refer to [5].

Theorem 2.5 [5] [17] *For every two integers n and k with $2 \leq k \leq n$,*

$$\kappa_{a+b}(K_{a,b}) = \left\lfloor \frac{ab}{a+b-1} \right\rfloor.$$

Note that the STP number is just $\lambda_n(G)$. Actually, it is just a simple case for $\lambda_k(G)$. S. Li, W. Li and X. Li [26] got the number of edge-disjoint spanning trees for complete 3-partite graphs.

Theorem 2.6 [26] *For every two integers n and k with $2 \leq k \leq n$,*

$$\kappa_{x+y+z}(K_{x,y,z}) = \left\lfloor \frac{xy + yz + zx}{x + y + z - 1} \right\rfloor.$$

From the above Theorems 2.5 and 2.6, a question is arisen: What is the number of edge-disjoint spanning trees of complete multipartite graphs? But this problem seems very difficult to solve. So they focused on complete equipartition multipartite graphs, and obtained the following theorem.

Theorem 2.7 [26] *For any two integers $a \geq 3$ and $b \geq 2$,*

$$\kappa_{ab}(K_b^a) = \left\lfloor \frac{a(a-1)b^2}{2(ab-1)} \right\rfloor.$$

One can see that, for complete graphs, complete bipartite graphs, complete 3-partite graphs and complete equipartition multipartite graphs, the number of edge-disjoint spanning trees of G is $\left\lfloor \frac{e(G)}{|V(G)|-1} \right\rfloor$. So they proposed the following conjecture.

Conjecture 2.8 [26] *For a complete multipartite graph G , the number of edge-disjoint spanning trees of G is*

$$\left\lfloor \frac{e(G)}{|V(G)| - 1} \right\rfloor.$$

All above results only consider the case $k = n$. Next, with a new method – *list method*, they computed the value of generalized k -connectivity of all complete bipartite graphs for $2 \leq k \leq n$.

Theorem 2.9 [17] *Given any two positive integers $a \leq b$, let $K_{a,b}$ denote a complete bipartite graph with a bipartition of sizes a and b , respectively. Then we have the following results: if $k > b - a + 2$ and $a - b + k$ is odd, then*

$$\kappa_k(K_{a,b}) = \frac{a + b - k + 1}{2} + \left\lfloor \frac{(a - b + k - 1)(b - a + k - 1)}{4(k - 1)} \right\rfloor$$

if $k > b - a + 2$ and $a - b + k$ is even, then

$$\kappa_k(K_{a,b}) = \frac{a + b - k}{2} + \left\lfloor \frac{(a - b + k)(b - a + k)}{4(k - 1)} \right\rfloor$$

and if $k \leq b - a + 2$, then

$$\kappa_k(K_{a,b}) = a$$

For a complete equipartition 3-partite graph, they got the following theorem.

Theorem 2.10 [26] *Given any two positive integers $b \geq 2$, let K_b^3 denote a complete 3-partite graph in which every part contains exactly b vertices. Then we have the following results:*

$$\kappa_k(K_b^3) = \frac{a + b - k + 1}{2} + \left\lfloor \frac{(a - b + k - 1)(b - a + k - 1)}{4(k - 1)} \right\rfloor$$

$$\kappa_k(K_b^3) = \begin{cases} \left\lfloor \frac{[k_2/3] + k_2 - 2kb}{2(k-1)} \right\rfloor + 3b - k, & \text{if } k \geq \frac{3b}{2}; \\ \left\lfloor \frac{3bk + 3b - k + 1}{2k + 1} \right\rfloor, & \text{if } \frac{3b}{4} < k \leq \frac{3b}{2} \text{ and } k \equiv 1 \pmod{3}; \\ \left\lfloor \frac{3bk + 6b - 2k + 1}{2k + 2} \right\rfloor, & \text{if } b \leq k < \frac{3b}{2} \text{ and } k \equiv 2 \pmod{3}; \\ \left\lfloor \frac{3b}{2} \right\rfloor, & \text{if } k < \frac{3b}{2} \text{ and } k \equiv 0 \pmod{3}; \\ \left\lfloor \frac{3b + 1}{2} \right\rfloor, & \text{otherwise.} \end{cases}$$

In a complete bipartite graph, if $k > b - a + 2$ and $a - b + k$ is odd, then $\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k-1}{2}})$, in part X there are $\frac{a-b+k-1}{2}$ vertices not in S , and in part Y there are $\frac{a-b+k-1}{2}$ vertices not in S . The number of vertices in each part but not in S is almost the same. And if $k > b - a + 2$ and $a - b + k$ is even, then $\kappa_k(K_{a,b}) = \kappa(S_{\frac{a-b+k}{2}})$, in part X there are $\frac{a-b+k}{2}$ vertices not in S , and in part Y there are $\frac{a-b+k}{2}$ vertices not in S . The number of vertices in each part but not in S is the same.

In a complete equipartition 3-partite graph, if $k = 0 \pmod{3}$, then $\kappa_k(K_b^3) = \kappa(S_{\frac{k}{3}, \frac{k}{3}, \frac{k}{3}})$, in part U there are $b - \frac{k}{3}$ vertices not in S , in part V there are $b - \frac{k}{3}$ vertices not in S , and in part W there are $b - \frac{k}{3}$ vertices not in S . The number of vertices in each part but not in S is the same. If $k = 1 \pmod{3}$, then $\kappa_k(K_b^3) = \kappa(S_{\frac{k+2}{3}, \frac{k-1}{3}, \frac{k-1}{3}})$. And if $k = 2 \pmod{3}$, then $\kappa_k(K_b^3) = \kappa(S_{\frac{k+1}{3}, \frac{k+1}{3}, \frac{k-2}{3}})$. In both cases, the number of vertices in each part but not in S is almost the same. So they proposed two conjectures.

Conjecture 2.11 [26] *For a complete equipartition a -partite graph G with partition (X_1, X_2, \dots, X_a) and integer $k = ab + c$, where b, c are integers and $0 \leq c \leq a - 1$, we have $\kappa_k(G) = \kappa(S)$, where S is a k -subset of $V(G)$ such that $|S \cap X_1| = \dots = |S \cap X_c| = b + 1$ and $|S \cap X_{c+1}| = \dots = |S \cap X_a| = b$.*

Conjecture 2.12 [26] *For a complete multipartite graph G , we have $\kappa_k(G) = \kappa(S)$, where S is a k -subset of $V(G)$ such that the number of vertices in each part but not in S are almost the same.*

3 Sharp bounds for generalized connectivity

Later in Section 7 we will know that it is almost impossible to get the exact value of the generalized (edge-) connectivity for a given arbitrary graph. So people aim to give some nice bounds for it, especially sharp upper and lower bounds.

From Theorems 2.3 and 2.4, i.e., $\kappa_k(K_n) = n - \lceil k/2 \rceil$ and $\lambda_k(K_n) = n - \lceil k/2 \rceil$, one can see the following two consequences since any connected graph G is a subgraph of a complete graph K_n .

Theorem 3.1 [21] *For a connected graph G of order n and $n \geq k \geq 3$, we have $1 \leq \kappa_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

Theorem 3.2 [21] *For a connected graph G of order n and $n \geq k \geq 3$, we have $1 \leq \lambda_k(G) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

For the above two theorems, one can easily check that the complete graph K_n attains the upper bound and the tree T_n attains the lower bound.

People mainly focus on sharp upper and lower bounds of $\kappa_k(G)$ and $\lambda_k(G)$ in terms of κ and λ , respectively. It seems difficult to get the sharp lower bound of $\kappa_k(G)$. So S. Li, X. Li and W. Zhou focused on the case that $k = 3$, and obtained the following theorem.

Theorem 3.3 [16] *Let G be a connected graph with n vertices. For every two integers s and r with $s \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\kappa(G) = 4s + r$, then $\kappa_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$. Moreover, the lower bound is sharp. We simply write $\kappa_3(G) \geq \frac{3\kappa-2}{4}$.*

The proof of Theorem 3.3 is interesting. At first, they considered to prove the theorem for the case that $\kappa(G) = 4k$, where $k \in \mathbb{N}^+$. The other cases can be verified similarly. Suppose that $S = \{v_1, v_2, v_3\}$. Since G is κ -connected, there are κ internally disjoint v_1v_2 -paths $P_1, P_2, \dots, P_\kappa$. Let $X = V(P_1 \cup \dots \cup P_\kappa)$.

Consider the simple case that v_3 is not in X . Obviously, $|X| \geq \kappa$ and so by the Fan Lemma there exists a κ -fan $\{M_1, M_2, \dots, M_\kappa\}$ from v_3 to X . If the terminal vertices $y_1, y_2, \dots, y_\kappa$ of $M_1, M_2, \dots, M_\kappa$ can be regarded as on the κ paths $P_1, P_2, \dots, P_\kappa$, respectively, then we may let $y_i \in P_i$ and let $T_i = M_i \cup P_i$ for $1 \leq i \leq \kappa$. So there are $\kappa(G) = 4k > 3k$ internally disjoint trees connecting S . Otherwise, there are at least two terminal vertices on a single v_1v_2 -path and without loss of generality, let $y_1, y_2 \in V(P_1)$ be such that y_1 is closer to v_1 than y_2 on P_1 . Then G has κ internally disjoint v_1v_2 -paths $P'_1 = v_1P_1y_1M_1v_3M_2y_2P_2v_2, P_2, \dots, P_\kappa$ and v_3 is on P'_1 . Thus, they converted this case to the one that v_3 is in X . For $v_3 \in X$, they gave a useful tool- the *Path-Bundle Transformation*.

Path-Bundle Transformation

Denote the connectivity $\kappa(G)$ of G simply by κ . Let $S = \{v_1, v_2, v_3\}$. For some integers t with $1 \leq t \leq \lfloor \frac{\kappa}{2} \rfloor$ and s with $s \geq t+1$, a family $\{P_1, P_2, \dots, P_s\}$ of s v_1v_2 -paths is called an *(s, t) -original-path-bundle connecting S* , if v_3 is on t paths P_1, P_2, \dots, P_t of them, and the s paths have no internal vertices in common except v_3 , as shown in Figure 3.1 (a).

If there is not only an (s, t) -original-path-bundle $\{P'_1, P'_2, \dots, P'_s\}$ connecting S , but also $\{M_1, M_2, \dots, M_{\kappa-2t}\}$ of $\kappa - 2t$ internally disjoint (v_3, X) -paths avoiding the vertices in $V(P'_1 \cup \dots \cup P'_t - \{v_1, v_2, v_3\})$, where $X = V(P'_{t+1} \cup \dots \cup P'_s)$, then we call the family of paths $\{P'_1, P'_2, \dots, P'_s\} \cup \{M_1, M_2, \dots, M_{\kappa-2t}\}$ an *(s, t) -reduced-path-bundle*

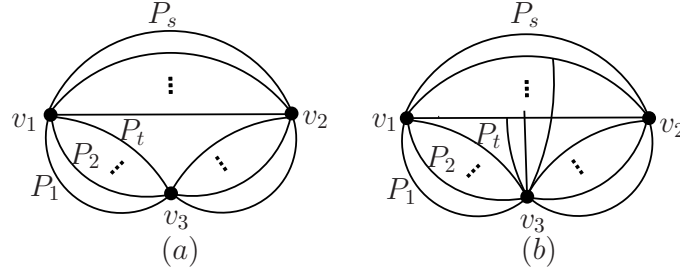


Figure 3.1 (a) An (s, t) -original-path-bundle; (b) An (s, t) -reduced-path-bundle and the illustration for the “Path-bundle Transformation”.

connecting S , as shown in Figure 3.1 (b).

They proved that if an (s, t) -original-path-bundle connecting S exists, then an (s, t) -reduced-path-bundle connecting S must exist. The operation of transforming an (s, t) -original-path-bundle connecting S to an (s, t) -reduced-path-bundle connecting S is called *the Path-Bundle Transformation*.

They sorted out two types of special (s, t) -reduced-path-bundle, which are two crucial structures in the proof of Theorem 3.3.

Type I: If $\{P'_1, P'_2, \dots, P'_s\} \cup \{M_1, M_2, \dots, M_{\kappa-2t}\}$ is a (κ, t) -reduced-path-bundle such that $1 \leq t \leq \lceil \frac{\kappa}{2} \rceil - 1$ and the terminal vertices $y_1, y_2, \dots, y_{\kappa-2t}$ of $M_1, M_2, \dots, M_{\kappa-2t}$ are on $\kappa-2t$ distinct paths of $P'_{t+1}, \dots, P'_\kappa$, then the (κ, t) -reduced-path-bundle is called *Type I*, as shown in Figure 3.2 (a). One can see that if G contains (κ, t) -reduced-path-bundle of Type I, then G contains $\kappa - \lceil \frac{t}{2} \rceil$ pairwise internally disjoint trees connecting S .

Type II: When κ is even and $t = \frac{\kappa}{2}$, then a (κ, t) -reduced-path-bundle is called *Type I*, as shown in Figure 3.2 (b). One can see that if G contains a (κ, t) -reduced-path-bundle of Type I, then G contains $\kappa - \lceil \frac{t}{2} \rceil$ pairwise internally disjoint trees connecting S .

By repeating the operation of the Path-Bundle Transformation, the procedure terminates either we get a (κ, t) -reduced-path-bundle of Type I or a $(\kappa, \frac{\kappa}{2})$ -reduced-path-bundle of Type II. Thus, $\kappa - \lceil \frac{t}{2} \rceil \geq 3k$, as required. The proof is complete.

To show that the lower bound of Theorem 3.3 is sharp, for $\kappa(G) = 4k + 2i$ with $i = 0$

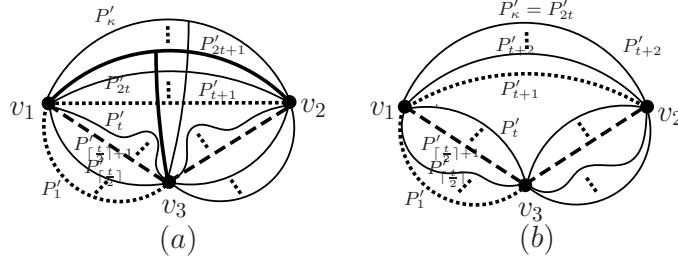


Figure 3.2 (a) An (κ, t) -reduced-path-bundle of Type I; (b) An (κ, t) -reduced-path-bundle of Type II.

or 1, they constructed a graph G as follows (see Figure 3.3 (a)): Let $Q = Y_1 \cup Y_2$ be a vertex cut of G , where Q is a clique and $|Y_1| = |Y_2| = 2k + i$, $G - Q$ has 2 components C_1, C_2 . $C_1 = \{v_3\}$ and v_3 is adjacent to every vertex in Q ; $C_2 = \{v_1\} \cup \{v_2\} \cup X$, $|X| = 2k + i$, the subgraph induced by X is an empty graph, each vertex in X is adjacent to every vertex in $Q \cup \{v_1, v_2\}$, v_i is adjacent to every vertex Y_i for $i = 1, 2$. It can be checked that $\kappa(G) = 4k + 2i$ and $\kappa_3(G) = 3k + i$, which means that G attains the lower bound.

For $\kappa(G) = 4k + 2i + 1$ with $i = 0$ or 1, they constructed a graph G as follows: Let $Q = Y_1 \cup Y_2 \cup \{y_0\}$ be a vertex cut of G , where Q is a clique and $|Y_1| = |Y_2| = 2k + i$. $G - Q$ has 2 components C_1, C_2 . $C_1 = \{v_3\}$ and v_3 is adjacent to every vertex in Q ; $C_2 = \{v_1\} \cup \{v_2\} \cup X$, $|X| = 2k + i$, the subgraph induced by X is an empty graph, each vertex in X is adjacent to every vertex in $Q \cup \{v_1, v_2\}$, v_i is adjacent to every vertex Y_i for $i = 1, 2$, and both v_1 and v_2 are adjacent to y_0 . It can be checked that $\kappa(G) = 4k + 2i + 1$ and $\kappa_3(G) = 3k + i + 1$, which means that G attains the lower bound.

Kriesell [8] gave a result on the Packing Steiner Tree Problem: Let $t \geq 1$ be a natural number and G a graph, and let $\{a, b, c\} \subseteq V(G)$ be $\lfloor \frac{8t+3}{6} \rfloor$ -edge-connected in G . Then there exists a system of t edge-disjoint $\{a, b, c\}$ -spanning trees. Using his this result, Li, Mao and Sun derived a sharp lower bound of $\lambda_3(G)$ and gave graphs attaining the bound. With this lower bound, they got some results for line graphs (see Section 5) and planar graphs.

Theorem 3.4 [20] *Let G be a connected graph with n vertices. For every two integers s and r with $s \geq 0$ and $r \in \{0, 1, 2, 3\}$, if $\lambda(G) = 4s + r$, then $\lambda_3(G) \geq 3s + \lceil \frac{r}{2} \rceil$.*

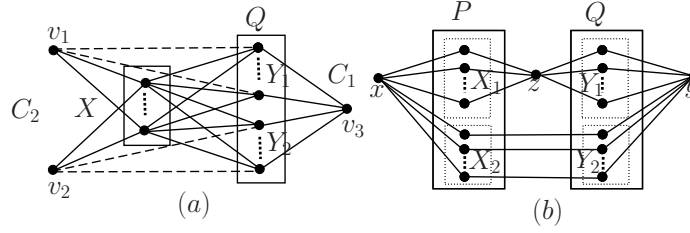


Figure 3.3 (a) For $\kappa(G) = 4k + 2i$ with $i = 0$ or 1 , the graph attaining the lower bound of Theorem 3.3. (b) The graph with $\lambda(G) = 4s$ and $\lambda_3(G) = 3s$ that attains the lower bound of Theorem 3.4.

Moreover, the lower bound is sharp. We simply write $\lambda_3(G) \geq \frac{3\lambda-2}{4}$.

To show that the lower bound is sharp, for $\lambda = 4s$ with $s \geq 1$ they constructed a graph G as follows (see Figure 3.3 (b)): Let $P = X_1 \cup X_2$ and $Q = Y_1 \cup Y_2$ be two cliques with $|X_1| = |Y_1| = 2s$ and $|X_2| = |Y_2| = 2s$. Let u, v be adjacent to every vertex in P, Q , respectively, and w be adjacent to every vertex in X_1 and Y_1 . Finally, they finished the construction of G by adding a perfect matching between X_2 and Y_2 . It can be checked that $\lambda = 4s$ and $\lambda(S) \geq 3s$. One can also check that for other three vertices of G the number of edge-disjoint trees connecting them is not less than $3s$. So $\lambda_3(G) = 3s$ and the graph G attains the lower bound.

For $\lambda = 4s + 1$, let $|X_1| = |Y_1| = 2s + 1$ and $|X_2| = |Y_2| = 2s$; for $\lambda = 4s + 2$, let $|X_1| = |Y_1| = 2s + 1$ and $|X_2| = |Y_2| = 2s + 1$; for $\lambda = 4s + 3$, let $|X_1| = |Y_1| = 2s + 2$ and $|X_2| = |Y_2| = 2s + 1$, where $s \geq 1$. Similarly, one can check that $\lambda_3(G) = 3s + 1$ for $\lambda = 4s + 1$; $\lambda_3(G) = 3s + 1$ for $\lambda = 4s + 2$; $\lambda_3(G) = 3s + 2$ for $\lambda = 4s + 3$.

For the case $s = 0$, we have $G = P_n$ such that $\lambda(G) = \lambda_3(G) = 1$; $G = C_n$ such that $\lambda(G) = 2$ and $\lambda_3(G) = 1$; $G = H_t$ such that $\lambda(G) = 3$ and $\lambda_3(G) = 2$, where H_t denotes the graph obtained from t copies of K_4 by identifying a vertex from each of them in the way shown in Figure 3.4.

Li, Mao and Sun gave a sharp upper bound of $\lambda_k(G)$.

Theorem 3.5 [21] *For any graph G of order n , $\lambda_k(G) \leq \lambda(G)$. Moreover, the upper bound is sharp.*

But, for $\kappa_k(G)$, S. Li proved that $\kappa_k(G) \leq \kappa(G)$ for $3 \leq k \leq 6$.

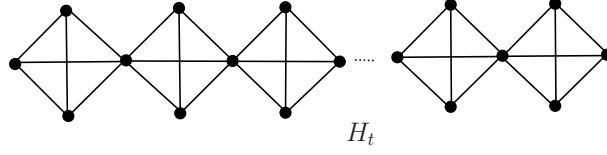


Figure 3.4 The graph H_t with $\lambda(H_t) = 3, \lambda_3(H_t) = 2$.

Theorem 3.6 [25] *Let G be a connected graph of order $n \geq 6$. Then for $3 \leq k \leq 6$, $\kappa_k(G) \leq \kappa(G)$. Moreover, the upper bound is always sharp for $3 \leq k \leq 6$.*

A natural question is why $\kappa_k(G) \leq \kappa(G)$ is not true for $k \geq 6$? One may want to solve this problem by proving $\kappa_k(G) \leq \kappa_{k-1}(G)$ for $3 \leq k \leq n$, namely, considering whether κ_k is monotonically decreasing in k . Unfortunately, S. Li found a counterexample G such that $\kappa_4(G) \geq \kappa_3(G)$. See the graph G shown in Figure 3 (a) for $i = 1$. S. Li showed that $\kappa(G) = 4k + 2$ and $\kappa_3(G) = 3k + 1$. It can be checked that the generalized 4-connectivity $\kappa_4(G) = 3k + 2$, which means that $\kappa_4(G) \geq \kappa_3(G)$ for a graph G .

She also gave a graph $H(k, t) = (K_{\frac{k}{2}} \cup K_{\frac{k}{2}}) \vee K_t$, where $k \geq 6$ and $t \geq 1$. Such a graph also indicates that the monotone properties of κ_k , namely, $\kappa_n \leq \kappa_{n-1} \leq \cdots \leq \kappa_4 \leq \kappa_3 \leq \kappa$ is not true for $2 \leq k \leq n$.

Theorem 3.7 [25] *For any two integer $k \geq 6$ and $t \geq 1$, $\kappa_{k+2}(H(k+1, t)) \geq \kappa_{k+1}(H(k, t))$.*

However, for cubic graphs the conclusion holds.

Theorem 3.8 [25] *If G is a cubic graph, then $\kappa_k(G) \leq \kappa_{k-1}(G)$ for $3 \leq k \leq n$.*

From Observation 2.1, we know that $\kappa_k(G) \leq \lambda_k(G) \leq \delta$. Actually, Li, Mao and Sun [21] showed that the graph $G = K_k \vee (n - k)K_1$ ($n \geq 3k$) satisfies that $\kappa_k(G) = \lambda_k(G) = \kappa(G) = \lambda(G) = \delta(G) = k$ (see Figure 1.1 (b)), which implies the upper bounds of Observation 2.1, Theorems 3.5 and 3.6. Li, Mao and Wang [21] gave a sufficient condition for $\lambda_k(G) \leq \delta - 1$. S. Li [25] obtained similar results on the generalized k -connectivity.

Theorem 3.9 [23] *Let G be a connected graph of order n with minimum degree δ . If there are two adjacent vertices of degree δ , then $\lambda_k(G) \leq \delta - 1$ for $3 \leq k \leq n$. Moreover, the upper bound is sharp.*

Theorem 3.10 [16] *Let G be a connected graph of order n with minimum degree δ . If there are two adjacent vertices of degree δ , then $\kappa_k(G) \leq \delta - 1$ for $3 \leq k \leq n$. Moreover, the upper bound is sharp.*

With the above bounds, we will on the applications. From Theorems 3.3 and 3.6, S. Li, X. Li and W. Zhou derived the sharp bounds for planar graphs.

Theorem 3.11 [16] *If G is a connected planar graph, then $\kappa(G) - 1 \leq \kappa_3(G) \leq \kappa(G)$.*

Motivated by constructing graphs to show that the upper and lower bounds are sharp, they obtained some lemmas. By the well-known Kuratowski's theorem [1], they verified the following lemma.

Lemma 3.12 [16] *For a connected planar graph G with $\kappa_3(G) = k$, there are no 3 vertices of degree k in G ,*

They also studied the generalized 3-connectivity of four kinds of graphs.

Lemma 3.13 [16] *If $\kappa(G) \geq 3$, then $\kappa_3(G - e) \geq 2$ for any edge $e \in E(G)$.*

Lemma 3.14 [16] *If G is a planar minimally 3-connected graph, then $\kappa_3(G) = 2$.*

Lemma 3.15 [16] *Let G be a 4-connected graph and let H be a graph obtained from G by adding a new vertex w and jointing it to 3 vertices of G . Then $\kappa_3(H) = \kappa(H) = 3$.*

Lemma 3.16 [16] *If G is a planar minimally 4-connected graph, then $\kappa_3(H) = 3$.*

If G is a connected planar graph, then $1 \leq \kappa(G) \leq 5$ by Theorem 3.11. Then, for each $1 \leq \kappa(G) \leq 5$, they gave some classes of planar graphs attaining the bounds of $\kappa_3(G)$, respectively.

Case 1: $\kappa(G) = 1$. For any graph G with $\kappa(G) = 1$, obviously $\kappa_3(G) \geq 1$ and so $\kappa_3(G) = \kappa(G) = 1$. Therefore, all planar graphs with connectivity 1 can attain the upper bound, but can not attain the lower bound.

Case 2: $\kappa(G) = 2$. Let G be a planar graph with connectivity 2 and having two adjacent vertices of degree 2. Then by Theorem, $\kappa_3(G) \leq 1$ and so $\kappa_3(G) \leq 1$, obviously $\kappa_3(G) \geq 1$ and so $\kappa_3(G) = 1 = \kappa(G) - 1$. Therefore, this class of graphs attain the upper bound.

Let G be a planar minimally 3-connected graph. By the definition, for any edge $e \in E(G)$, $\kappa_3(G - e) = 2$. Then by Theorem 3.5 and Lemma 3.13, it follows that $\kappa_3(G - e) = 2$. Therefore, the 2-connected planar graph $G - e$ attains the upper bound.

Case 3: $\kappa(G) = 3$. For any planar minimally 3-connected graph G , we know that $\kappa(G) = 3$ and by Lemma 3.14, $\kappa_3(G) = 2 = \kappa(G) - 1$. So this class of graphs attain the lower bound.

Let G be a planar 4-connected graph and let H be a graph obtained from G by adding a new vertex w in the interior of a face for some embedding of G and joining it to 3 vertices on the boundary of the face. Then H is still planar and by Lemma 3.15, one can immediately get that $\kappa_3(H) = 2 = \kappa(H) = 3$, which means that H attains the upper bound.

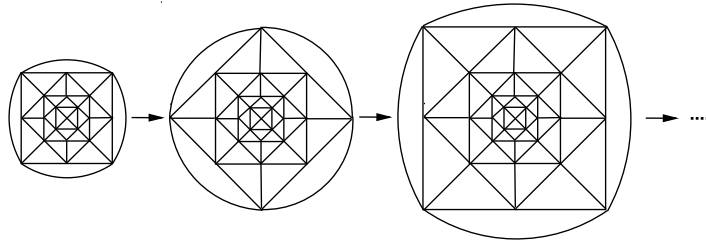


Figure 3.5 The graphs for the upper bound of Case 4

Case 4: $\kappa(G) = 4$. For any planar minimally 4-connected graph G , one know that $\kappa(G) = 4$, and by Lemma 3.16, $\kappa_3(G) = 3 = \kappa(G) - 1$. So this class of graphs attain the lower bound.

For every graph in Figure 3.5, the vertex in the center has degree 4 and it can be checked that for any 2 vertices there always exist four pairwise internally disjoint paths connecting them, which means that $\kappa(G) = 4$. It can also be checked that for any 3 vertices there always exist four pairwise internally disjoint paths connecting them. Combining this with Theorem 3.5, one can get that $\kappa_3 = 4$. Therefore, the graphs attain the upper bound. Moreover, we can construct a series of graphs according to

the pattern of Figure 3, which attain the upper bound.

Case 5: $\kappa(G) = 5$. For any planar graph G with $\kappa(G) = 5$, by Lemma 3.12 one can get that $\kappa_3(G) = 4$. So, any planar graph G with connectivity 5 can attain the lower bound, but obviously can not attain the upper bound.

Similarly, the following theorem is obvious by Theorems 3.4 and 3.5.

Theorem 3.17 [21] *If G be a connected planar graph, then $\lambda(G) - 1 \leq \lambda_3(G) \leq \lambda(G)$.*

4 Graphs with large generalized connectivity

From the last section, we know that $1 \leq \kappa_k(G) \leq n - \lceil k/2 \rceil$ and $1 \leq \lambda_k(G) \leq n - \lceil k/2 \rceil$ for a connected graph G . Li, Mao and Sun [21] considered to characterize graphs attaining the upper bounds, namely, graphs with $\kappa_k(G) = n - \lceil k/2 \rceil$ or $\lambda_k(G) = n - \lceil k/2 \rceil$. Since a complete graph K_n possesses the maximum generalized (edge-) connectivity, they wanted to find out the critical value of the number of edges, denoted by \mathfrak{t} , such that the generalized (edge-) connectivity of the resulting graph will keep being $n - \lceil k/2 \rceil$ by deleting \mathfrak{t} edges from a complete graph K_n but will not keep being $n - \lceil k/2 \rceil$ by deleting $\mathfrak{t} + 1$ edges. By further investigation, they conjectured that \mathfrak{t} may be 0 for k even and \mathfrak{t} may be $\frac{k-1}{2}$ for k odd.

First, they noticed that for arbitrary $S \subseteq V(G)$ there are two types of edge-disjoint trees connecting S : the tree of Type *I* is the tree whose edges all belong to $E(G[S])$; the tree of Type *II* is the tree containing at least one edge of $E_G[S, W]$, where $W = V(G) \setminus S$. We denote the set of the edge-disjoint trees of Type *I* and Type *II* by \mathcal{T}_1 and \mathcal{T}_2 , respectively. Let $\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2$.

Lemma 4.1 [21] *Let $S \subseteq V(G)$, $|S| = k$ and T be a tree connecting S . If $T \in \mathcal{T}_1$, then T uses $k - 1$ edges of $E(G[S]) \cup E_G[S, W]$; If $T \in \mathcal{T}_2$, then T uses at least k edges of $E(G[S]) \cup E_G[S, W]$.*

They found that $|E(G[S]) \cup E_G[S, W]|$ is fixed once the graph G is given whatever there exist how many trees of Type *I* and how many trees of Type *II*. From Lemma 4.1, each tree will use certain number of edges in $E(G[S]) \cup E_G[S, W]$. Deleting excessive edges from a complete graph K_n will result in that the remaining edges in $E(G[S]) \cup E_G[S, W]$ will not form $n - \lceil k/2 \rceil$ trees. By using such an idea, they proved that

$\lambda_k(G) < n - \lceil \frac{k}{2} \rceil$ for $\mathfrak{t} \geq 1$ (k is even) and $\lambda_k(G) < n - \lceil \frac{k}{2} \rceil$ for $\mathfrak{t} \geq \frac{k+1}{2}$ (k is odd). Furthermore, from Observation 2.1, $\kappa_k(G) < n - \lceil \frac{k}{2} \rceil$ for $\mathfrak{t} \geq 1$ (k is even) and $\kappa_k(G) < n - \lceil \frac{k}{2} \rceil$ for $\mathfrak{t} \geq \frac{k+1}{2}$ (k is odd).

Next, they only need to find out $n - \lceil k/2 \rceil$ internally disjoint trees connecting S in G , where $G = K_n$ for k even; $G = K_n \setminus M$ and M is an edge set such that $|M| = \frac{k}{2}$ for k odd. Obviously, it only needs to consider the case that k is odd. But the difficulty is that the edges of $E(G[S]) \cup E_G[S, W]$ belong to a tree connecting S and can not be wasted as many as possible. Fortunately, Nash-Williams-Tutte theorem provides a perfect solution. They first derived the following lemma from Theorem 1.1.

Lemma 4.2 [21] *If n is odd and M is an edge set of the complete graph K_n such that $0 \leq |M| \leq \frac{n-1}{2}$, then $G = K_n \setminus M$ contains $\frac{n-1}{2}$ edge-disjoint spanning trees.*

They wanted to find out $\frac{k-1}{2}$ edge-disjoint spanning trees in $G[S]$ (By the definition of internally disjoint trees, these trees are internally disjoint trees connecting S , as required). Then their basic idea is to seek for some edges “evenly” in $G[S]$, and let them together with the edges of $E_G[S, W]$ form $n - k$ internally disjoint trees. Applying a procedure designed by them, they proved that there are indeed $n - k$ internally disjoint trees in the premise that $G[S]$ contains $\frac{k-1}{2}$ edge-disjoint spanning trees. Actually, Lemma 4.2 can guarantee the existence of such $\frac{k-1}{2}$ trees. Then they found out $n - \frac{k-1}{2}$ internally disjoint trees connecting S and accomplish the proof of the following theorem.

Theorem 4.3 [21] *For a connected graph G of order n and $n \geq k \geq 3$, $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

Combining Theorem 4.3 and Observation 2.1, they obtained the following theorem for $\lambda_k(G)$.

Theorem 4.4 [21] *For a connected graph G of order n and $n \geq k \geq 3$, $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$ if and only if $G = K_n$ for k even; $G = K_n \setminus M$ for k odd, where M is an edge set such that $0 \leq |M| \leq \frac{k-1}{2}$.*

With Nash-Williams-Tutte Theorem, they luckily characterized graphs attaining the upper bound. But one can see that it is difficult to characterize graphs with $\kappa_k(G) = n - \lceil \frac{k}{2} \rceil - 1$. So H. Li, X. Li, Y. Mao and Y. Sun considered the case that $k = 3$, namely, they considered graphs with $\kappa_3(G) = n - 3$.

Theorem 4.5 [22] *Let G be a connected graph of order n ($n \geq 3$). Then $\kappa_3(G) = n - 3$ if and only if G is a graph obtained from the complete graph K_n by deleting an edge set M such that $K_n[M] = P_4$ or $K_n[M] = P_3 \cup P_2$ or $K_n[M] = C_3 \cup P_2$ or $K_n[M] = rP_2$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$).*

From the above, one can see that there are many same structures and properties between generalized connectivity and generalized edge-connectivity. But, for the edge case, Li, Mao and Wang [23] showed that the statement is different.

Theorem 4.6 [23] *Let G be a connected graph of order n . Then $\lambda_3(G) = n - 3$ if and only if G is a graph obtained from the complete graph K_n by deleting an edge set M such that $K_n[M] = rP_2$ ($2 \leq r \leq \lfloor \frac{n}{2} \rfloor$) or $K_n[M] = P_4 \cup sP_2$ ($0 \leq s \leq \lfloor \frac{n-4}{2} \rfloor$) or $K_n[M] = P_3 \cup tP_2$ ($0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor$) or $K_n[M] = C_3 \cup tP_2$ ($0 \leq t \leq \lfloor \frac{n-3}{2} \rfloor$).*

5 Results for graph operations

In this section we will survey the results for complementary graphs, line graphs, and graph Cartesian products.

5.1 For complementary graphs and line graphs

Alavi and Mitchem in [29] considered Nordhaus-Gaddum-type results for connectivity and edge-connectivity. Li, Mao and Sun [21] were concerned with analogous inequalities involving the generalized k -connectivity and generalized k -edge-connectivity.

Theorem 5.1 [21] *For any graph G of order n , we have $1 \leq \kappa_k(G) + \kappa_k(\overline{G}) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

The same Nordhaus-Gaddum-type result as that for the generalized k -connectivity can be established for the generalized k -edge-connectivity, which is stated as follows:

Theorem 5.2 [23] *For any graph G of order n , we have $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$. Moreover, the upper and lower bounds are sharp.*

One can easily check that the complete graph K_n attains the upper bound and the complete bipartite graph $K_{1,n-1}$ attains the lower bound.

Chartrand and Steeart investigated the relation between the connectivity and edge-connectivity of a graph and its line graph. They proved that if G is a connected graph, then (1) $\kappa(L(G)) \geq \lambda(G)$ if $\lambda(G) \geq 2$; (2) $\lambda(L(G)) \geq 2\lambda(G) - 2$; (3) $\kappa(L(L(G))) \geq 2\kappa(G) - 2$. With the help of Theorem 3.4, Li, Mao and Sun also considered the generalized 3-connectivity and 3-edge-connectivity for line graphs.

Theorem 5.3 [21] *If G is a connected graph, then*

- (1) $\lambda_3(G) \leq \kappa_3(L(G))$.
- (2) $\lambda_3(L(G)) \geq \frac{3}{2}\lambda_3(G) - 2$.
- (3) $\kappa_3(L(L(G))) \geq \frac{3}{2}\kappa_3(G) - 2$.

Moreover, the bounds are sharp.

First, they proved (1) of Theorem 5.3. Next, combining Theorem 3.4 with (1) of Theorem 5.3, they derived (2) and (3) of Theorem 5.3. One can check that the bounds are sharp if we take $G = C_n, K_6, K_6$, respectively, for (1), (2), (3).

Let $L^0(G) = G$ and $L^1(G) = L(G)$. Then for $k \geq 2$, the k -th iterated line graph $L^k(G)$ is defined by $L(L^{k-1}(G))$. The next statement follows immediately from Theorem 5.3 and a routine application of induction.

Corollary 5.4 [21] *If $\kappa_3(G) = p$, then $\lambda_3(L^k(G)) \geq (\frac{3}{2})^k(p - \frac{8}{3}) + 4$ and $\kappa_3(L^k(G)) \geq (\frac{3}{2})^{k-1}(p - \frac{8}{3}) + 4$.*

5.2 For Cartesian products

The Cartesian product of graphs is an important method to construct a bigger graph, and plays a key role in design and analysis of networks. In the past several decades, many authors have studied the (edge-) connectivity of the Cartesian product graphs. Specially, Sabidussi in [34] derived the following perfect and well-known theorem on the connectivity of Cartesian product graphs: for two connected graphs G and H , $\kappa(G \square H) \geq \kappa(G) + \kappa(H)$.

In [21], H. Li, X. Li and Y. Sun studied the generalized 3-connectivity of Cartesian product graphs. Their results could be seen as a generalization of Sabidussi's result. As usual, in order to get a general result, they first began with a special case.

Theorem 5.5 [20] *Let G be a graph and P_m be a path with m edges. We have the following results.*

- (1) *If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G \square P_m) \geq \kappa_3(G)$. Moreover, the bound is sharp.*
- (2) *If $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G \square P_m) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp.*

Note that $Q_n \cong P_2 \square P_2 \square \cdots \square P_2$, where Q_n is the n -hypercube. They got the following corollary.

Corollary 5.6 [20] *Let Q_n be the n -hypercube with $n \geq 2$. Then $\kappa_3(Q_n) = n - 1$.*

Example 5.7 *Let H_1 and H_2 be two complete graphs of order n , and let $V(H_1) = \{u_1, u_2, \dots, u_n\}$, $V(H_2) = \{v_1, v_2, \dots, v_n\}$. We now construct a graph G as follows:*

$$V(G) = V(H_1) \cup V(H_2) \cup \{w\}, \text{ where } w \text{ is a new vertex};$$

$$E(G) = E(H_1) \cup E(H_2) \cup \{u_i v_j | 1 \leq i, j \leq n\} \cup \{w u_i | 1 \leq i \leq n\}.$$

It is easy to check that $\kappa_3(G \square K_2) = \kappa_3(G) = n$ by Theorems 2.3 and 2.9.

They showed the bounds of (1) and (2) in Theorem 5.5 are sharp by Example 5.7 and Corollary 5.6.

Next, they studied the 3-connectivity of the Cartesian product of a graph G and a tree T , which will be used in Theorem 5.9.

Theorem 5.8 [20] *Let G be a graph and T be a tree. We have the following results.*

- (1) *If $\kappa_3(G) = \kappa(G) \geq 1$, then $\kappa_3(G \square T) \geq \kappa_3(G)$. Moreover, the bound is sharp.*
- (2) *If $1 \leq \kappa_3(G) < \kappa(G)$, then $\kappa_3(G \square T) \geq \kappa_3(G) + 1$. Moreover, the bound is sharp.*

The bounds of (1) and (2) in Theorem 5.8 are sharp by Example 5.7 and Corollary 5.6.

They mainly investigated the 3-connectivity of the Cartesian product of two connected graphs G and H . By decomposing H into some trees connecting 2 vertices or 3 vertices, they considered the Cartesian product of a graph G and a tree T and obtained Theorem 5.8 by Theorem 5.9.

Theorem 5.9 [20] *Let G be a graph and T be a tree. We have the following results.*

(1) *If $\kappa(G) > \kappa_3(G)$, then $\kappa_3(G \square H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

(2) *If $\kappa_3(G) = \kappa(G)$, then $\kappa_3(G \square H) \geq \kappa_3(G) + \kappa_3(H)$. Moreover, the bound is sharp.*

They also showed that the bounds of (1) and (2) in Theorem 5.9 are sharp. Let K_n be a complete graph with n vertices, and P_m be a path with m vertices, where $m \geq 2$. Since $\kappa_3(P_m) = 1$ and $\kappa_3(K_n) = n - 2$, it is easy to see that $\kappa_3(K_n \square P_m) = n - 1$. Thus, $K_n \square P_m$ is a sharp example for (1). For (2), Example 5.7 is a sharp one.

6 Extremal problems

In this section, we survey the results on the extremal problems of generalized connectivity and generalized edge-connectivity.

6.1 The minimal size of a graph with $\kappa_3 = 2$

In [25], S. Li focused on the following problem: given any positive integer $n \geq 4$, is there a smallest integer $f(n)$ such that every graph of order n and size $e(G) \geq f(n)$ has $\kappa_3 \geq 2$? She proved that every graph G of order n and size $e(G) = \frac{n^2}{2} - \frac{3n}{2} + 3$ can be regarded as a graph obtained from K_n by deleting $n - 3$ edges. Therefore, $f(n) \leq \frac{n^2}{2} - \frac{3n}{2} + 3$. On the other hand, let G be a graph obtained from K_{n-1} by deleting a vertex v and joining v to any one vertex of K_{n-1} . Clearly, the order is n and the size is $\frac{n^2}{2} - \frac{3n}{2} + 3$. But $\kappa_3(G) \geq \delta(G) = 1$. So $f(n) > \frac{n^2}{2} - \frac{3n}{2} + 3$. Thus, the following theorem is easily seen.

Theorem 6.1 [25] *Given any positive integer $n \geq 4$, there exists a smallest integer $f(n) = \frac{n^2}{2} - \frac{3n}{2} + 3$ such that every graph G of order n and size $e(G) \geq f(n)$ has $\kappa_3(G) \geq 2$.*

S. Li, X. Li and Y. Shi [18] determined the minimal number of edges of a graph with $\kappa_3(G) \geq 2$, i.e., for a graph G of order n and size $e(G)$ with $\kappa_3(G) = 2$, that is

Theorem 6.2 [18] *If G is a graph of order n with $\kappa_3(G) = 2$, then $e(G) \geq \frac{6}{5}n$. Moreover, the lower bound is sharp for all $n \geq 4$ but $n = 9, 10$, whereas the best lower bound for $n = 9, 10$ is $\lceil \frac{6}{5}n \rceil + 1$.*

They constructed a graph class to show that the bound of Theorem 6.2 is sharp: for a positive integer $t \neq 2$, let $C = x_1y_1x_2y_2 \cdots x_{2t}y_{2t}x_1$ be a cycle of length $4t$. Add t new vertices z_1, z_2, \dots, z_t to C , and join z_i to x_i and x_{i+t} , for $1 \leq i \leq t$. The resulting graph is denoted by H . Then, the generalized 3-connectivity of H is 2, namely, $\kappa_3(H) = 2$ (see Figure 6.1).

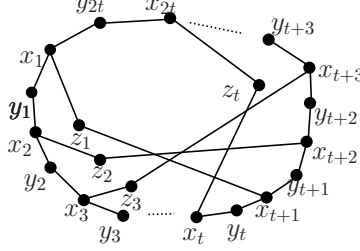


Figure 6.1 The graph H with $\kappa_3(H) = 2$

Though it is easy to find the sharp lower bound of $e(G)$, a little progress has been made on the sharp upper bound. So S. Li phrased an open problem as follows.

Open Problem: Let G be a graph of order n and size $e(G)$ such that G is minimal for $\kappa_3 = 2$. Find the sharp upper bounds $g(n)$ of $e(G)$.

She considered the complete bipartite graph $G = K_{2,n-2}$. It can be easily checked that the size of G is $2(n-2)$ and $\kappa_3(G) = 2$. Since every edge is incident with a vertex of degree 2, the graph is minimal for $\kappa_3 = 2$. Therefore, $g(n) \geq 2n - 4$. On the other hand, by Lemma 3.13, if $\kappa(G) \geq 3$, then $\kappa_3(G - e) \geq 2$ for any edge $e \in E(G)$. The following results may provide a little information on $g(n)$.

Lemma 6.3 [25] *Let G be a minimally k -connected graph of order n . If $n \geq 3k - 2$, then $e(G) \leq k(n - k)$.*

Since any graph of $\kappa_3 = 2$ can be obtained from a minimally 3-connected graph by deleting some edges, by Lemma 6.3 the size of a graph of $\kappa_3 = 2$ and order $n \geq 7$ is less than $3(n - 3) - 1$. So, $g(n) \leq 3n - 10$. Thus, $2n - 4 \leq g(n) \leq 3n - 10$. But she did not determine the exact value of $g(n)$.

6.2 Generalized maximal local connectivity

Put $\bar{\kappa}(G) = \max\{\kappa(x, y) \mid x, y \in V(G), x \neq y\}$, where $\kappa(x, y)$ is the maximum number of internally disjoint paths connecting x and y in G . Put $f(n; \bar{\kappa} \leq \ell) =$

$\max\{e(G_n) : \bar{\kappa}(G) \leq \ell\}$. The problem of $f(n; \bar{\kappa} \leq \ell)$ is called the problem of maximal local connectivity. The first problem of this kind was considered by Bollobás [35]. For $\bar{\kappa}(G) \leq \ell$, $f(n; \bar{\kappa} \leq \ell) \geq \lfloor \frac{\ell+1}{2} \rfloor (n-1)$. Since $f(n; \bar{\kappa} \leq \ell) = \lfloor \frac{\ell+1}{2} \rfloor (n-1)$ for $\ell = 2, 3$, Bollobás and Erdős conjectured that the equality holds, but their conjecture was disproved by Leonard [37] for $k = 4$, and then Mader [38] constructed graphs disproving it for every $k \geq 4$. There are many results on this topic, we refer to [35, 36, 37, 38, 39].

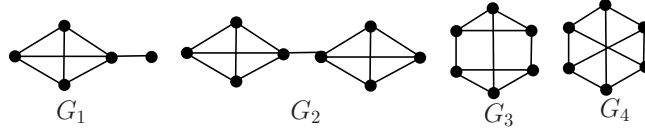


Figure 6.2 The graphs for Theorem 6.4

H. Li, X. Li and Y. Mao considered a similar problem on the generalized connectivity. Put $\bar{\kappa}_k(G) = \max\{\kappa(S) | S \subseteq V(G), |S| = k\}$, where $\kappa(S)$ is the maximum number of internally disjoint trees connecting S in G . Put $f(n; \bar{\kappa}_k \leq \ell) = \max\{e(G_n) : \bar{\kappa}_k(G) \leq \ell\}$. For $\ell = 2$, they determined the exact values of $f(n; \bar{\kappa}_3 \leq 2)$ and characterized graphs attaining these values.

Theorem 6.4 *Let G be a connected graph of order n such that $\bar{\kappa}_3(G) \leq 2$. Then*

$$e(G) \leq \begin{cases} 2n - 2 & \text{if } n = 4, \\ 2n - 3 & \text{if } n \neq 4. \end{cases}$$

with equality if and only if $G \in \mathcal{E}_n^$.*

For $n = 3$ and $n = 4$, $\mathcal{E}_n^* = K_n$. For $n \geq 5$, they gave a graph class $\mathcal{E}_n = \{H_n^1, H_n^2, H_n^3, H_n^4, H_n^5, H_n^6, H_n^7\}$ (see Figure 6.2). $G \in \mathcal{E}_n^*$ if and only if $G \in \{G_1, G_2, G_3, G_4\}$ (see Figure 6.3) or $G \in \mathcal{E}_n$ or there exists $G' \in \mathcal{E}_r$ such that G can be obtained from G' by identifying a clique of order 4 with some vertices of degree 2 in G' .

Remark: Let n, ℓ be odd, and G' be a graph obtained from a $(\ell - 2)$ -regular graph of order $n - 2$ by deleting an edge, and $G = G' \vee K_2$. Then $\delta(G) = n - 2$, $\bar{\kappa}_3(G) \leq \ell$ and $e(G) = \frac{\ell+2}{2}(n+1) - \lfloor \frac{\ell}{2} \rfloor - 1$.

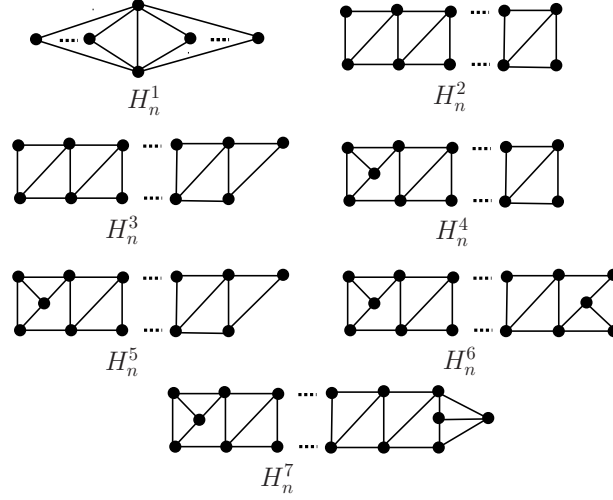


Figure 6.3 The graph class \mathcal{E}_n

Otherwise, let G' be a $(\ell - 2)$ -regular graph of order $n - 2$ and $G = G' \vee K_2$. Then $\delta(G) = n - 2$, $\overline{\kappa}_3(G) \leq \ell$ and $e(G) = \frac{\ell+2}{2}(n+1) - \lfloor \frac{\ell}{2} \rfloor$. Therefore,

$$f(n; \overline{\kappa}_3 \leq \ell) \geq \begin{cases} \frac{\ell+2}{2}(n+1) - \lfloor \frac{\ell}{2} \rfloor - 1 & \text{if } n, \ell \text{ are odd,} \\ \frac{\ell+2}{2}(n+1) - \lfloor \frac{\ell}{2} \rfloor & \text{if otherwise.} \end{cases}$$

7 Algorithm and complexity

As it is well known, for any graph G , we have polynomial-time algorithm to get the connectivity $\kappa(G)$. A natural question is whether there is a polynomial-time algorithm to get the $\kappa_k(G)$. First, S. Li, X. Li and W. Zhou investigated κ_3 in [16].

7.1 Results for κ_3

For a graph G , by the definition of $\kappa_3(G)$, it is natural to study $\kappa(S)$ first, where S is a 3-subset of $V(G)$. A question is then raised: for any positive integer k , given a 3-subset S of $V(G)$, is there a polynomial-time algorithm to determine whether $\kappa(S) \geq k$? they gave a positive answer by converting the problem into the k -Linkage Problem [28].

Theorem 7.1 [16] *Given a fixed positive integer k , for any graph G the problem of deciding whether G contains k internally disjoint trees connecting $\{v_1, v_2, v_3\}$ can be*

solved by a polynomial-time algorithm, where v_1, v_2, v_3 are any 3 vertices of $V(G)$.

From this theorem together with $\kappa_3(G) = \min\{\kappa(S)\}$, the following theorem can be easily obtained.

Theorem 7.2 [16] *Given a fixed positive integer k , for any graph G the problem of deciding whether $\kappa_3(G) \geq k$ can be solved by a polynomial-time algorithm.*

The following two corollaries are immediate.

Corollary 7.3 [16] *Given a fixed positive integer k , for any graph G with connectivity κ , the problem of deciding whether $\kappa_3(G) \geq k$ can be solved by a polynomial-time algorithm.*

Corollary 7.4 [16] *Given a fixed positive integer δ , for any graph G with connectivity δ , the problem of deciding whether $\kappa_3(G) \geq k$ can be solved by a polynomial-time algorithm.*

For a planar graph, they proved the following theorem.

Theorem 7.5 [16] *For a planar graph G with connectivity $\kappa(G)$, the problem of determining $\kappa_3(G)$ has a polynomial-time algorithm and its complexity is bounded by $O(n^8)$.*

They mentioned that the above complexity is not very good, and so the problem of finding a more efficient algorithm was an interesting one.

The complexity of the problem of determining $\kappa_3(G)$ for a general graph is not known: can it be solved in polynomial time or \mathcal{NP} hard? Nevertheless, from Theorems 3.3 and 3.6, they derived a polynomial-time algorithm to determine it approximately with a constant ratio.

Theorem 7.6 [16] *The problem of determining $\kappa_3(G)$ for any graph G can be solved by a polynomial-time approximation algorithm with a constant ratio about $\frac{3}{4}$.*

7.2 Results for general κ_k

In [19], S. Li and X. Li generalized the result of Theorem 7.1.

Theorem 7.7 [16] *For two fixed positive integers k_1 and k_2 , given a graph G and a k_1 -subset S of $V(G)$, the problem of deciding whether G contains k_2 internally disjoint trees connecting S can be solved by a polynomial-time algorithm.*

By Theorem 7.1, the following theorem can be easily obtained.

Theorem 7.8 [19] *Given two fixed positive integers k_1 and k_2 , for any graph G the problem of deciding whether $\kappa_{k_1}(G) \geq k_2$ can be solved by a polynomial-time algorithm.*

Actually, Theorem 7.8 is a generalization of Theorem 7.2. For one of k_1 and k_2 not fixed, S. Li and X. Li proposed two problems.

Problem 7.9 [19] *Given a graph G , a 4-subset S of $V(G)$ and an integer $2 \leq k \leq n - 1$, decide whether there are k internally disjoint trees connecting S , namely decide whether $\kappa(S) \geq k$?*

Problem 7.10 [19] *Given a graph G , a subset S of $V(G)$, decide whether there are two internally disjoint trees connecting S , namely decide whether $\kappa(S) \geq 2$?*

First, they proved that Problem 7.9 is \mathcal{NP} -complete by reducing 3-DM to it. Next they showed that for a fixed $k_1 \geq 5$, in Problem 7.9 replacing the 4-subset of $V(G)$ with a k_1 -subset of $V(G)$, the problem is still \mathcal{NP} -complete, which can be proved by reducing Problem 1 to it. Thus, they obtained the following theorem.

Theorem 7.11 [19] *For any fixed integer $k_1 \geq 4$, given a graph G , a k_1 -subset S of $V(G)$ and an integer $2 \leq k_2 \leq n - 1$, deciding whether there are k_2 internally disjoint trees connecting S , namely deciding whether $\kappa(S) \geq k_2$, is \mathcal{NP} -complete.*

By reducing 3-SAT to Problem 7.10, they also verified that Problem 7.10 is \mathcal{NP} -complete. Next they showed that for a fixed integer $k \geq 3$, in Problem 7.10 if we want to decide whether there are k internally disjoint trees connecting S rather than two, the problem is still \mathcal{NP} -complete, which can be easily proved by reducing Problem 7.10 to it. Then they got the following theorem.

Theorem 7.12 *For any fixed integer $k \geq 2$, given a graph G and a subset S of $V(G)$, deciding whether there are k internally disjoint trees connecting S , namely deciding whether $\kappa(S) \geq k$, is \mathcal{NP} -complete.*

As shown in Theorems 7.7 and 7.8, S. Li and X. Li [18] only showed that for any fixed integer $k_1 \geq 4$, given a graph G , a k_1 -subset S of $V(G)$ and an integer $2 \leq k_2 \leq n - 1$, deciding whether $\kappa(S) \geq k_2$ is \mathcal{NP} -complete. For $k_1 = 3$, the complexity is yet not known. So, S. Li conjectured that it is \mathcal{NP} -complete.

Conjecture 7.13 [25] *Given a graph G and a 3-subset S of $V(G)$ and an integer $2 \leq k \leq n - 1$, deciding whether there are k internally disjoint trees connecting S , namely deciding whether $\kappa(S) \geq k$, is \mathcal{NP} -complete.*

From Theorem 7.8, given two fixed integers $k_1 \geq 4$ and k_2 , for any graph G and a k_1 -subsets S of $V(G)$ the problem of deciding whether $\kappa(S) \geq k_2$ can be solved by a polynomial-time algorithm, and since $\kappa_{k_1}(G) = \min\{\kappa(S)\}$, where the minimum is taken over all k_1 -subsets S of $V(G)$ (such S 's are polynomially many), they can also obtain that the problem of deciding whether $\kappa_{k_1}(G) \geq k_2$ can be solved by a polynomial-time algorithm.

Therefore, when k_1 is a fixed integer, the problem of deciding whether $\kappa_{k_1}(G) \geq k_2$ is not harder than that of deciding whether $\kappa(S) \geq k_2$, where S is a k_1 -subset of $V(G)$.

By Theorem 7.11, we know that if k_2 is not a fixed positive integer, the problem of deciding whether $\kappa(S) \geq k_2$ is \mathcal{NP} -complete, but the complexity of the problem of deciding whether $\kappa_{k_1}(G) \geq k_2$ is still not known. So S. Li proposed the following conjecture.

Conjecture 7.14 [25] *For a fixed integer $k_1 \geq 3$, given a graph G and an integer $2 \leq k \leq n - 1$, the problem of deciding whether $\kappa_{k_1}(G) \geq k_2$, is \mathcal{NP} -complete.*

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